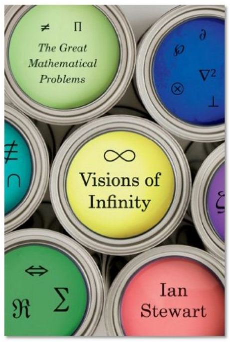
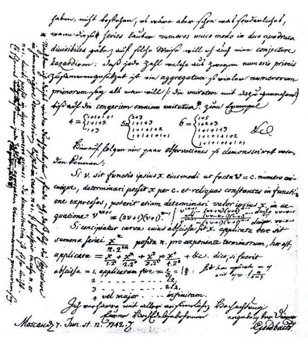
Visions of infinity. The Great Mathematical Problems, by *Ian Stewart*. Basic Books, 2013, ISBN 978-04-6502-240-3 (hbk), 352 pp.



Ian Stewart

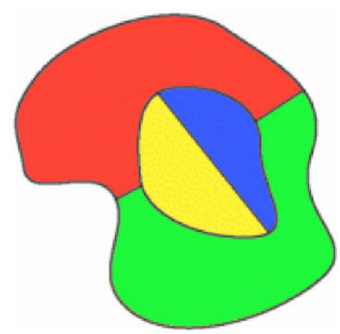
Ian Stewart is a professor of mathematics at Warwick University, well known for his many books on popular science. In this recent book he introduces the reader to the *Great Problems of Mathematics* (with capitals!). These are the ones that were often formulated as conjectures that gained fame because they remained unsolved for many years or that are still not answered in the positive or negative and that are still worth a million dollar. Since these usually generated a lot of mathematical progress, the answer whether the conjecture is true or false is not really the most important objective, but the new proof techniques and the whole new body of mathematics needed becomes a driving force for the development of mathematics and the true reason why prestigious prizes are rewarded for their proofs. And we all know several of these like Fermat's last theorem and the four colour problem. You're too late if you want to start solving those but the Riemann conjecture, and the P/NP problem for example are still open.

It is the intention of the author to describe the origin, the history, and the development of some of these really big questions that have driven mathematics in new directions, hence illustrating their importance for mathematics and science in general. It is not easy, but Stewart at least tries, with some success I must say, to transfer the ideas without the plethora of formulas and difficult mathematical concepts that one would expect. What counts are the notions, not the notation as Gauss once remarked. Obviously some of the more technical details have to be skipped, and while the ideas of the early chapters are easy to transfer, it becomes much more difficult, if one is not familiar with some more advanced mathematics, to grasp the millennium problems of the Clay Mathematics Institute that are introduced in the later chapters.



Goldbach's letter to Euler

π transcendental



4 colours are necessary

So after an introductory chapter, Stewart takes us on a journey along the history of the great challenges that mathematics lived through, or that are still marinating, waiting for a solution. Most enlightening is to see how he illustrates that a breakthrough of some tough problem is triggered by some advance in an, at first sight, unrelated piece of mathematics. Stewart's advise: if you can't solve it, put it aside, do something else, and come back later.

The first topic is the *Goldbach conjecture* (1742): every integer $n > 2$ is the sum of 2 primes. Everybody believes it to be true, but it remains unproven. Stewart takes his time to introduce prime numbers, their history and their computation. This pays because several of the subsequent problems are related to prime numbers as well. He explains what has been shown and what is not, how it has been linked to the Riemann conjecture and the open problems left for the present and future generations.

Squaring the circle is a problem that dates back to the Greeks and is clearly linked to π. This is a nice illustration of how an obviously geometric problem, is reformulated as an algebraic one, solving polynomial equations, leading to (complex) analysis, when one wants to prove that π is irrational and transcendental. Much more recent is the *four colour problem*. It was first formulated by a South African law student in 1852. He asked his brother's advise. The brother was studying mathematics at the University College in Londen where he asked A. De Morgan and the latter exposed it to the math community. Again, the solution is not important for cartography because there are many other reasons to choose a colour for the map, but it started research is networks and graphs, and it has been generalized to colour problems on much more complex topological surfaces. It was finally proved in 1976 by Appel and Haken and it revolutionized the concept of a proof, since it was the first time that a proof relied on the verification by a computer of many cases to which the problem was reduced. Too many to be checked by humans.

Another famous problem is known as the *Kepler conjecture*. Although the original version of 1611 by Kepler appeared in a booklet on 6-pointed snowflakes, it is related to sphere packing basically asking how dense equal spheres can be packed. Every grocer knows how to mount oranges in a pyramid, but it took 387 years for a proof to be found. Again, a computer was needed to solve the global optimization problem. A formal proof avoiding the computer is still an ongoing project. The *Mordell conjecture* (1922) was proved by Faltings in 1983. It is about Diophantine equations but has a geometric formulation stating that a curve of genus $g > 1$ over $\mathbb{Q}$ has only finitely many rational points. Stewart uses this on his path towards *Fermat's last theorem*, since one may start from Pythagoras equation $x^2 + y^2 = z^2$ with integers to the equation of a circle $(x/z)^2 + (y/z)^2 = 1$ with rational numbers and subsequently to a generalization where the circle is replaced by an elliptic curve. Stewart's account of the human aspects of doubt and fear of failure or err when Wiles was working on his proof of FLT has some remote reminiscences of a mathematical thriller.

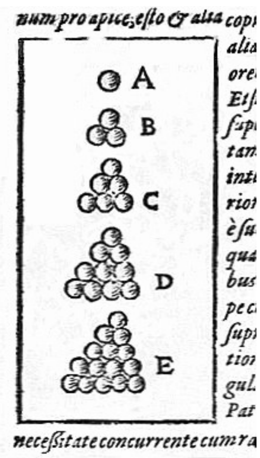
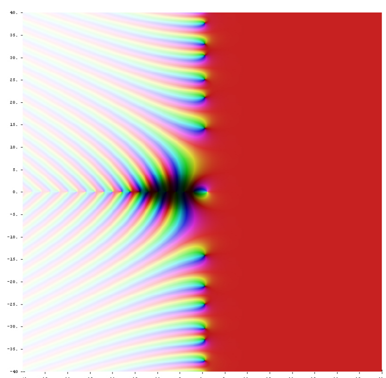


image from Kepler's snowflake book

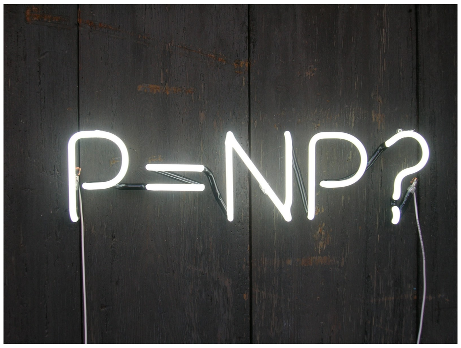


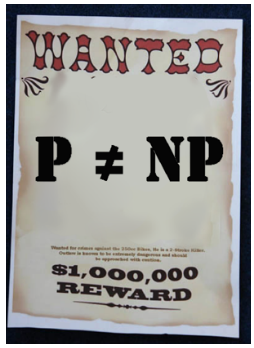
Riemann ζ function

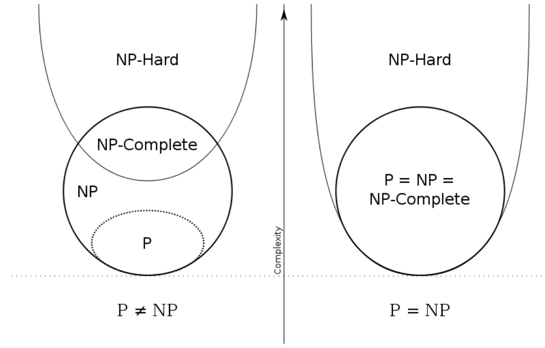
Although Poincaré got the award of the Swedish King Oscar II in 1887, he did tot really solve the originally posed *three body problem* that was suggested by Mittag-Leffler. Nowadays there are numerical techniques to solve equations with a chaotic solution approximately, rigorous proofs and many questions remain unanswered. The answers are directly related to fundamental questions about the stability of our solar system.

Back to prime number distribution with the *Riemann hypothesis* (1859). Again number theory is lifted to complex analysis in the study of the ζ-function (Stewart needs quite some pages to come to this point). It is explained how this leads to the conjecture that the zeros of the ζ-function are on the critical line $x = 1/2$ in the complex plane. This is the most famous open problem in mathematics today. It survived Hilbert's 1900 unsolved problems and it is reformulated as one of the Clay millennium problems. Most mathematicians believe it to be true as numerics seem to indicate but a proof is still missing.

The other six millennium problems are the subject of the following 6 chapters, in which the mathematics that Stewart needs become tougher. But where the going gets tough, the tough get going. The *Poincaré conjecture* (1904) was solved by Perelman in 2002, but because it took 8 years for the math community to verify his proof, the eccentric Perelman, totally disappointed with that situation, has withdrawn from mathematics and refuses all contact with the media. He declined the EMS prize (1996), Fields Medal (2006) and the Clay millennium prize (2010).

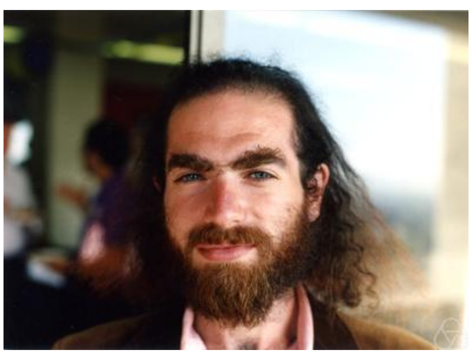






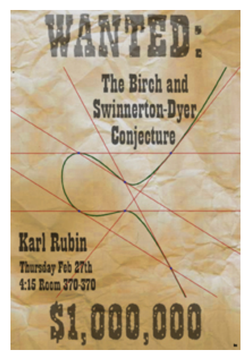
The *P/NP problem* is still open and the outcome is uncertain: are hard problems such as the traveling salesman problem solvable with polynomial time algorithms? The answer to this question seems to be NP-hard itself. P stands for polynomial time complexity, NP means non-deterministic polynomial time complexity, and NP-complete are the hardest problems from class NP. If one can solve these, then any other NP problem can be solved in polynomial time.

Solving the *Navier-Stokes equation* is a problem from applied mathematics. Can one verify that the small changes made by numerical procedures don't miss some turbulent solution because the approximation is not fine enough. In January 2014, Otelbaev claims to have solved this problem. The proof is at the moment (March 2014) still under dispute but probably flawed. The *mass gap hypothesis* relates to quantum field theory of elementary particles. These quantum particles have a nonzero lower bound for their mass even though the waves travel at the speed of light. In relativity theory, the mass would be zero. The *Birch-Swinnerton-Dyer conjecture* is another millennium problem about rational solutions of certain elliptic curve equations. Finally the *Hodge conjecture* connects topology, algebra, geometry and analysis to be able to say something about algebraic cycles on projective algebraic varieties.



G. Perelman (1993)

M. Otelbaev



BSD conjecture

Although Stewart tries very hard to introduce the unprepared reader to the problems and the techniques for the latter four problems, the much more advanced mathematical needs make these chapters much harder to read than the earlier ones. As a conclusion, he gives his own opinion of what will and what will not be proved in the (near) future. Just in case the reader gave up on the Riemann hypothesis and is looking for inspiration to find another really challenging problem, Stewart provides a list of 12 somewhat less known open questions that are as yet unsolved.

Stewart's entertaining style, his meticulous sketching of the historical context, his sharp analysis of the importance of the problems and their consequences, his broad insight on the wide spectrum of mathematics and his understanding of the human behind the mathematician, struggling for solutions and recognition, each one with his own character, makes this book, like his other books, a very interesting read. Some of the less mathematically experienced readers may not reach the end, but whatever can be assimilated is worthwhile.

Adhemar Bultheel